The Binomial Theorem

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1 Introduction

The Binomial Theorem is used to expand binomials, that is, brackets consisting of two distinct terms. The formula for the Binomial Theorem is as follows:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + b^n\end{aligned}$$

Note that we could write $\binom{n}{0}$ and $\binom{n}{n}$ as the coefficients for a^n and b^n respectively, but these coefficients are equal to one so we have omitted them.

One will need to know this formula, or at least know how to derive it on the spot. Another formula which is very useful and **should be remembered**, is this (derived at the end):

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + ...$$

2 Basic intuition

Consider the expression $(a+b)(c+d)$. How might we multiply these brackets together? Hopefully, you should be able to expand this quite naturally:

$$(a+b)(c+d) = ac + ad + bc + bd$$

What have we actually done here? We have 'picked' one term from the first bracket, then multiplied it by a term we 'picked' from the second bracket. Above, the ac term is made from picking a from the first bracket and c from the second bracket, then multiplying.

We then repeat this for all possible ways of 'picking' terms, then add them up to give the final expansion. The result of this is that 'everything is multipled by everything'; the final expression consists of terms each formed by the multiplication of exactly one component from each bracket.

This bit of intuition is very important, and will later explain why the binomial coefficient, $\binom{n}{k}$ (or ${}^nC_k$ if you prefer) shows up.

3 Some simple expansions

3.1 The expanision of $(1+x)^2$

Let us begin with one of the simplest possible binomials, $(1+x)^2$. Expanding $(1+x)^2$ like we did with $(a+b)(c+d)$, we get $1+x+x+x^2$ (which is then simplified to $1+2x+x^2$).

How do we explain this result with regards to 'picking' terms? Remember, we can pick one term from the first bracket and one term from the second bracket (it is implicit that we then multiply). In the final expansion:

- The 1 comes from picking a 1 from each bracket.
- The x terms come from picking an x from a bracket and picking a 1 from the other.
- The x^2 term comes from picking an x from both brackets.

There are two x terms because there are two possible brackets from which we could pick the x: we can either pick the x from the first bracket, or from the second bracket.

Generally, **there is more than one way we can produce a certain power of** x. These need to be accounted for.

3.2 The expanision of $(1+x)^3$

If we write out the brackets fully, we have $(1+x)(1+x)(1+x)$. We know that the expansion will have a constant, an x, an x^2, and an x^3 term. The trouble is that we don't know the coefficients (the constant numbers in front of each power of x).

We will therefore say, for now, that:

$$(1+x)^3 = \_ + \_x + \_x^2 + \_x^3$$

(We just need to fill in the blanks!)

- It is clear that the constant term is formed as a result of picking 1 from each of the brackets. Since $1 \times 1 \times 1 = 1$, this coefficient is clearly 1.

- How do we get the x term? Now things are getting interesting. Clearly, from the three brackets, we need to pick one x and two 1's, because $x \times 1 \times 1 = x$. The thing is, there are three possible brackets we can choose our x from: but we only need one! So how many ways can we choose one x and two 1's? At this point, you must rememer your combinatorics. We have three, and we need to choose one (order doesn't matter). The answer is clearly $\binom{3}{1}$.
- What about the x^2 term? Applying the same reasoning as above, the x^2 term is formed from multiplying two x terms and a 1: $x \times x \times 1 = x^2$. How many ways, from the three brackets, can we pick two x terms? The answer is simply $\binom{3}{2}$.
- The x^3 term obviously comes from multiplying three x terms together, i.e we need to choose an x from each of the three brackets. There is only one way we can do this, so the last coefficient is 1.

Putting this all together and computing the binomial coefficients, we arrive at the expansion:

$$\begin{aligned}(1+x)^3 &= 1 + \binom{3}{1}x + \binom{3}{2}x^2 + x^3 \\ &= 1 + 3x + 3x^2 + x^3\end{aligned}$$

Problem 1
What is the coefficient of x^4 in the expansion of $(1+x)^7$?
Solution:
Questions asking for specific coefficients are common, and often underpin the solutions to more complex binomial questions. **It is thus critical that you know how to do this type of question**.
Imagine writing out all the brackets next to each other, like this:

$$(1+x)(1+x)(1+x)(1+x)(1+x)(1+x)(1+x)$$

The x^4 term will be made by multiplying four x terms together (and obviously three remaining 1's). How many ways can we choose four x terms, remembering that there are a total of seven to choose from? Simple. The binomial coefficient will be $\binom{7}{4}$.

3.3 A slight modification to the problem

In the above problem, what if I had asked for the coefficient of x^4 in the expansion of $(2+x)^7$? How would the problem change? Very little, it turns out. I am going to rewrite the above explanation, with some minor amendments.

The x^4 term will be made by multiplying four x terms together (and obviously three remaining 2's). How many ways can we choose four x terms, remembering that there are a total of seven to choose from? Simple. The binomial coefficient will be $\binom{7}{4}$. Remember that after 'picking' everything, we have to multiply (this has been implicit in everything we've done). But let's explicitly do this now.

We picked four x terms and three 2 terms, so we consider $x \times x \times x \times x \times 2 \times 2 \times 2$, which is written more compactly as $(2)^3(x)^4$. But before we finish, we must remember to multiply by the binomial coefficient.

The final x^4 term is therefore $\binom{7}{4}(2)^3x^4$, so the coefficient is $\binom{7}{4}(2)^3$.

4 Applying your knowledge...

Problem 2
Find the coefficient of x^5 in the expansion of $(3x-2)^8$.
Solution:
Out of 8 brackets, we pick five $3x$'s and three of the (-2) terms. The binomial coefficient is thus '8 choose 5', or $\binom{8}{5}$. Combining all this, we know that the x^5 term will be $\binom{8}{5}(-2)^3(3x)^5$. Ask your calculator for the answer; it should say -108864.

Problem 3
Find the coefficient of a^5b^7 in the expansion of $(a+b)^{12}$.
Solution:
We need to choose five a terms out of the twelve brackets, so the binomial coefficient is $\binom{12}{5}$. Note that we could instead say $\binom{12}{7}$; both of these coefficients are equal (can you think why?).

Problem 4
Find the term containing x^{10} in the expansion of $(6+2x^2)^7$.
Solution:
This looks nasty, but actually isn't. To make x^{10}, we need to choose five $2x^2$ terms (since x^2 multiplied by itself five times gives x^{10}). The binomial coefficient is thus $\binom{7}{5}$ Hence the term in question will be $\binom{7}{5}(6)^2(2x^2)^5$. You can quickly check for yourself that expanding this will give you the required x^{10} term.

Problem 5
Fully expand $(a+b)^6$
Solution:
There will be seven terms in this expansion. How do I know? Well, the first term will be made from choosing six a's. The next term will be made from choosing five a's and one b. The third term will be made from choosing four a's and two b's. And so on. Hence, the expansion is:

$$a^6 + \binom{6}{5}a^5b + \binom{6}{4}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{2}a^2b^4 + \binom{6}{1}ab^5 + b^6$$

I doubt that the IB is mean enough to ask you to do this just for the lols, but it may help you for the next question.

Problem 6
Determine the constant term in the expansion of $(x + \frac{1}{x^2})^6$.
Solution:
Now is the time to use your brain! This is a typical problem that comes up in both SL and HL papers. Both terms in our binomial contain x, so the constant term is certainly **not** x^6. We need to find some particular choice of terms that will allow the x's to cancel. Easier said than done.

What happens if we choose two x terms and four $\frac{1}{x^2}$ terms? Well, the binomial coefficient is clearly $\binom{6}{2}$. Thus, the term in the expansion will be $\binom{6}{2}(x)^2(\frac{1}{x^2})^4$ or $\binom{6}{2}\frac{x^2}{x^8}$. Simplifying using the laws of exponents, we get $\frac{15}{x^6}$. We wanted x^0, but instead got x^{-6}. This means that we chose too many of the $\frac{1}{x^2}$ terms.

What about having three x terms and three $\frac{1}{x^2}$ terms? The binomial coefficient will just be $\binom{6}{3}$. The associated term in the expansion will be $\binom{6}{3}(x)^3(\frac{1}{x^2})^3$, which simplifies to $\frac{20}{x^3}$. We wanted x^0, but instead got x^{-3}. Too negative still.

We will now make a more sensible choice: four x terms and two $\frac{1}{x^2}$ terms. The binomial coefficient is $\binom{6}{4}$. Thus, the term in the expansion will be $\binom{6}{4}(x)^4(\frac{1}{x^2})^2$ or $\binom{6}{4}\frac{x^4}{x^4}$. Something special happens here. The x's all cancel out, leaving a lone $\binom{6}{4}$. The constant term is thus 15.

To summarise, finding the constant term is a bit of a balancing act, you need to choose just the right number of x^a terms such that they cancel out with the $\frac{1}{x^b}$ terms. This involves a small bit of guesswork, which can be done mentally quite fast.

Problem 7
What is the constant term in the expansion of $(3x^2 - \frac{1}{x})^9$?
Solution:
Practice makes perfect – let's try this again. We immediately know that we are going to need more of the $\frac{1}{x}$ terms, because each $3x^2$ term has two x's in it. At this point, you have to think of the possiblities. If I take two $3x^2$ terms and seven $(-\frac{1}{x})$ terms, the final result will have x^{-5}. Too low. If I take three $3x^2$ terms and six $(-\frac{1}{x})$ terms, oh look, the result will have x^0.

Hence the term we need is $\binom{9}{3}(3x^2)^3(-\frac{1}{x})^6$. As we discussed, the x's will cancel out. The numbers, however, will still remain. Thus the coefficient will be $\binom{9}{3} \times 3^3 \times (-1)^6$.

Problem 8
Find the x^2 term in the expansion of $(2 + x)^4(1 + x)^7$
Solution:
Please don't expand everything, unless you want to amuse an HL maths student standing behind you. What you need to do is find the first few terms (up to x^2) in each of the

brackets. I trust you'll be able to do this by now.

$$(2+x)^4(1+x)^7 = (16+32x+24x^2+\ldots)(1+7x+21x^2+\ldots)$$

Now, we think. We need an x^2 – how can we make this? Remember, when multiplying brackets, 'everything is multiplied by everything'. So, somewhere in the expansion, the 16 from the first bracket will be multiplied by the 1, or the $7x$, or the $21x^2$ (and so on). The question is, which of these will yield us an x^2? Obviously, it will be $16\times 21x^2$.

We now look at the next term in the first bracket, the $32x$. Again, this will be multiplied by every term in the second bracket – but which one gives us an x^2? Obviously, $32x\times 7x$.

Proceeding, we examine the $24x^2$. With which term in the second bracket should we multiply this to result in an x^2? Well, we multiply it by the constant term since $24x^2$ already contains an x^2. Thus, $24x^2\times 1$.

To summarise:

$$\begin{aligned} 16\times 21x^2 &= 336x^2 \\ 32x\times 7x &= 224x^2 \\ 24x^2\times 1 &= 24x^2 \end{aligned}$$

That is to say, when we multiply out $(2+x)^4(1+x)^7$, the expansion will contain the above terms (and many others with different powers of x).

$$(2+x)^4(1+x)^7 = \ldots + 336x^2 + 224x^2 + 24x^2 + \ldots$$

But don't just stand there, simplify! All of these x^2 terms can be combined to give $584x^2$. Hence, the coefficient of x^2 is 584. Quite easily done.

Here is the final compiled solution, which is what I'd write in an exam:

$$\begin{gathered} (2+x)^4(1+x)^7 = (16+32x+24x^2+\ldots)(1+7x+21x^2+\ldots) \\ 16\times 21x^2 = 336x^2 \\ 32x\times 7x = 224x^2 \\ 24x^2\times 1 = 24x^2 \\ \therefore \text{ the coefficient of } x^2 \text{ is } 336+224+24 = 584. \end{gathered}$$

Extension Problem
Find the first four terms in the expansion of $\sqrt{1+x}$
Solution:
Unlikely to come up in SL, but you never know. We start by finding a general formula for the expansion of $(1+x)^n$, then later substitute in $n=\frac{1}{2}$.

The Binomial Theorem states that:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$$

(We need only consider the first four terms.)
Since we want the expansion of $(1+x)^n$, we substitute $a = 1$ and $b = x$. Thus:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + x^n$$

As promised, we now just substitute in $n = \frac{1}{2}$. But if you try finding $\binom{0.5}{1}$, your calculator will return an error message. Fair enough: how can we choose one thing from a selection of 0.5 things – the question makes no sense! We will delay substituting $n = \frac{1}{2}$, and instead try to find nicer expressions for the combinatorial coefficients $\binom{n}{1}$, $\binom{n}{2}$, and $\binom{n}{3}$. Recall the actual formula for $\binom{n}{r}$:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Notice that this can be written differently, by 'expanding' the factorials.

$$\frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots}{r!(n-r)(n-r-1)(n-r-2)\cdots}$$

Both the top and bottom of the fraction contain $(n-r)(n-r-1)\ldots$ terms, which can be cancelled. Thus:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

If we substitute $r = 1, 2, 3\ldots$ respectively:

$$\binom{n}{1} = \frac{n!}{(n-1)!} = n$$
$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$
$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

This derives the very useful formula for the general expansion of $(1+x)^n$, which was stated in the introduction, and works for non-integer values of n. Now, and only now, may we substitute $n = \frac{1}{2}$.

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{1/2(-1/2)}{2}x + \frac{1/2(-1/2)(-3/2)}{3!}x^2 + \ldots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\end{aligned}$$

This is actually quite a remarkable result. The square root of $1 + x$ can be expressed as a never-ending polynomial. Isn't life great?